



Extending precolourings of circular cliques

Richard C. Brewster^{*}, Jonathan A. Noel

Department of Math and Stats, Thompson Rivers University, PO Box 3010, Kamloops, BC, Canada, V2C 5N3

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ABSTRACT

Let G be a graph with circular chromatic number $\chi_c(G) = \frac{k}{q}$. Given $P \subseteq V(G)$ where each component of $G[P]$ is isomorphic to the circular clique $G_{k,q}$, suppose the vertices of P have been precoloured with a (k', q') -colouring. We examine under what conditions one can be assured the precolouring extends to the entire graph. We study sufficient conditions based on $\frac{k'}{q'} - \frac{k}{q}$ as well as the distance between precoloured components of $G[P]$. In particular, we examine a conjecture of Albertson and West showing the conditions for extendibility are more complex than anticipated in their work.

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1. Introduction

In [6], Thomassen asks if given a planar graph G and a 5-colouring of $P \subseteq V(G)$ with the restriction that the vertices of P are far enough apart, can the colouring be extended to the entire graph? In [1], Albertson answers the question in the affirmative provided the precoloured vertices are at a distance of at least four from each other. Fundamental in the result are two key components: First, one requires 5 colours for the extended colouring despite $\chi(G) \leq 4$. (In general, a precolouring of a planar graph is not extendible to a 4-colouring of the entire graph.) Second, the precoloured vertices must be sufficiently far apart. In fact, Albertson proves the following.

Theorem 1 (Albertson [1]). *Suppose $\chi(G) = r$ and $P \subseteq V(G)$ such that the distance between any two vertices in P is at least 4. Any $(r + 1)$ -colouring of P can be extended to an $(r + 1)$ -colouring of G .*

There are now many papers on colouring extensions. The introduction of [3] provides a nice overview. We focus on the situation where the precoloured vertices induce a collection of cliques.

Given $P \subseteq V(G)$, define $d(P)$ to be the minimum distance in G between any two connected components of $G[P]$. A k -colouring of P is called a k -precolouring. A more general result proved by Albertson in [1] says that any $(r + 1)$ -precolouring of an r -colourable graph can be extended to the entire graph where the set P of precoloured vertices consists of k -cliques provided $d(P) \geq 6k - 2$. This distance is improved by Kostochka to $d(P) \geq 4k$, (as cited and proved in [2]); and by Albertson and Moore to $d(P) \geq 3k$ when $r = k$, [2].

In [3], Albertson and West study extending circular colourings where the precoloured vertices form an independent set. They conjecture in the case that G is (k, q) -colourable and the precoloured components are circular cliques $G_{k,q}$, there is a distance d^* such that any (k', q') -precolouring can be extended provided $\frac{k'}{q'} - \frac{k}{q} \geq 1$ and $d(P) \geq d^*$.

We settle this conjecture in the negative by constructing an infinite family of graphs with the following property. Given any ℓ , there exists k, q and k', q' with $\frac{k'}{q'} - \frac{k}{q} \geq \ell$ for which the following holds: for any integer d , there is a graph G with circular chromatic number k/q , and which has two circular cliques $G_{k,q}$ at distance d such that some (k', q') -precolouring of the two cliques is nonextendible.

^{*} Corresponding author.

E-mail addresses: rbrewster@tru.ca (R.C. Brewster), jonathan-noel@mytru.ca (J.A. Noel).

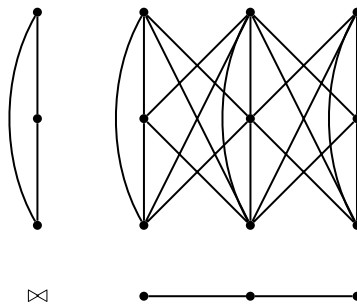


Fig. 1. The product $K_3 \bowtie P_3$ together with the two factors. The projection onto the first coordinate is a homomorphism to K_3 .

We then turn our attention to positive results. Extendibility results are established for the cases when one of the q' or q equals 1, i.e. one of the two colourings is a classical vertex colouring.

1.1. Circular colourings and the extension product

We provide some key definitions for our work here and refer the reader to [7] for standard definitions. Let k, q be positive integers such that $k \geq 2q$. A (k, q) -colouring of a graph G is an assignment $c : V(G) \rightarrow \{0, 1, 2, \dots, k-1\}$ such that for $uv \in E(G)$, $q \leq |c(u) - c(v)| \leq k - q$. The *circular complete graph* or *circular clique* $G_{k,q}$ has vertices $\{0, 1, \dots, k-1\}$ and edges $\{ij : q \leq |i - j| \leq k - q\}$. Thus $G_{k,1}$ is simply the (classical) complete graph on k vertices.

The circular complete graphs play the role in circular colourings as do the complete graphs in classical colourings. Adopting the homomorphism point of view, see [4,5], G admits a (k, q) -colouring if, and only if, there is a homomorphism $f : G \rightarrow G_{k,q}$. Recall, a *homomorphism* $f : G \rightarrow H$ is a mapping $f : V(G) \rightarrow V(H)$ such that $uv \in E(G)$ implies $f(u)f(v) \in E(H)$. We write $G \rightarrow H$ to indicate the existence of a homomorphism.

It turns out that $G_{k,q} \rightarrow G_{k',q'}$ if, and only if, $\frac{k}{q} \leq \frac{k'}{q'}$. Thus, given a graph G , if $G \rightarrow G_{k,q}$, then $G \rightarrow G_{k',q'}$ for any $\frac{k'}{q'} \geq \frac{k}{q}$. The *circular chromatic number* of a graph G is defined as

$$\chi_c(G) = \inf \left\{ \frac{k}{q} : G \rightarrow G_{k,q} \right\}.$$

In [4], Bondy and Hell show the infimum may be replaced by a minimum. The proof depends on the fact that optimum colourings must be surjective. Surjective mappings play a key role in our constructions of nonextendible families.

A fundamental construction in our work comes from [3].

Definition 1. Let G and H be graphs. The *extension product* $G \bowtie H$ has as its vertex set $V(G) \times V(H)$ with $(g_1, h_1)(g_2, h_2)$ an edge if $g_1g_2 \in E(G)$ and either $h_1h_2 \in E(H)$ or $h_1 = h_2$.

Alternatively one may view $G \bowtie H$ as the categorical product of G with a reflexive copy (a loop on each vertex) of H . It is straightforward to verify $G \bowtie H \rightarrow G$ via the projection onto the first coordinate (see Fig. 1). As a point of notation, for a fixed $i \in V(H)$, the subgraph induced by $\{(v, i) | v \in V(G)\}$ is isomorphic to G and is denoted by G^i . Given a homomorphism $f : G \bowtie H \rightarrow X$, the mapping defined by $f_i(u) := f(u, i)$ is a homomorphism $f_i : G^i \rightarrow X$. Given $\varphi : G \rightarrow X$, we may view $V(G) = V(G^i)$ and thus write $f_i = \varphi$ if $f_i(u) = \varphi(u)$ for all $u \in V(G)$.

Of particular importance for us is the product $G \bowtie P_n$. Recall P_n is the path of length $n - 1$ with vertex set $\{1, 2, \dots, n\}$.

2. Nonextendibility for large $\frac{k'}{q'} - \frac{k}{q}$

In this section we provide a negative answer to the conjecture of Albertson and West. We examine extending circular colourings of $G_{k,q} \bowtie P_n$ where the first and last copy of the circular clique $G_{k,q}$ have been (k', q') -coloured.

We begin with our key definition.

Definition 2. Let $\varphi : G \rightarrow H$ be a homomorphism and let $v \in V(G)$. The homomorphism φ is *uniquely extendible at v* if whenever $g : G \rightarrow H$ is a homomorphism with $g(u) = \varphi(u)$ for all $u \neq v$, then $g(v) = \varphi(v)$. If φ is uniquely extendible at v for all $v \in V(G)$, we simply say φ is *uniquely extendible*.

The following proposition provides a straightforward, but useful characterization of uniquely extendible homomorphisms.

Proposition 2. Let $\varphi : G \rightarrow H$ be a homomorphism. Then φ is uniquely extendible at v if, and only if,

$$\left| \bigcap_{w \in N_G(v)} N_H(\varphi(w)) \right| = 1.$$

Proof. Suppose φ is uniquely extendible at v . Let $T = \bigcap_{w \in N_G(v)} N_H(\varphi(w))$. The existence of φ ensures $|T| \geq 1$. If $x, y \in T$ with $\varphi(v) \neq y$, then we define the homomorphism g by

$$g(u) = \begin{cases} \varphi(u) & \text{if } u \neq v \\ y & \text{otherwise} \end{cases}$$

contrary to the assumption that φ is uniquely extendible.

On the other hand, if φ is not uniquely extendible at v , then there is a homomorphism g such that φ and g agree on all vertices of $G - v$ but $\varphi(v) \neq g(v)$. Then $\{\varphi(v), g(v)\} \subseteq \bigcap_{w \in N_G(v)} N_H(\varphi(w))$. The result follows. \square

The crux of our approach is captured in the following proposition.

Proposition 3. Let $\varphi : G \rightarrow H$ be uniquely extendible at v for all $v \in V(G)$. Let $f : G \bowtie P_n \rightarrow H$ be a homomorphism such that $f_1 = \varphi$. Then $f_i = \varphi$ for all $i, 1 \leq i \leq n$.

Proof. Consider a vertex $(v, 1)$ in $G \bowtie P_n$. The subgraph $T = G_1 - (v, 1) + (v, 2)$ is isomorphic to G . Moreover, the homomorphism f restricted to T is a homomorphism of a copy of G to H which agrees with φ on all vertices of $T - (v, 2)$. Thus $f(v, 2) = \varphi(v, 1)$. The result follows. \square

The following theorem demonstrates the existence of uniquely extendible homomorphisms between circular cliques.

Theorem 4. Let q and t be positive integers. Given any $k \equiv \pm 1 \pmod{t}$ such that $k \geq 3qt + 1$, there exists a uniquely extendible homomorphism $\varphi : G_{k,qt+1} \rightarrow G_{k,q+1}$.

Proof. Suppose k, q , and t satisfy the hypotheses. Further suppose $k + 1 = tg$ for some integer g . The case $k - 1 = tg$ is analogous. Denote the vertices of $G_{k,qt+1}$ by $\{v_i : 0 \leq i \leq k - 1\}$ and the vertices of $G_{k,q+1}$ by $\{i : 0 \leq i \leq k - 1\}$.

Define $\varphi : G_{k,qt+1} \rightarrow G_{k,q+1}$ by $\varphi(v_i) = gi \pmod{k}$. We claim φ is a homomorphism. First observe k and g are relatively prime; thus, the mapping φ is injective. Let $i \in V(G_{k,q+1})$. Consider $\varphi(v_{ti}) = gti = (k + 1)i = i \pmod{k}$. By injectivity, v_{ti} is the unique vertex mapping to i . Consider a nonneighbour of i , say $i + j$ for some $j \in \{\pm 1, \pm 2, \dots, \pm q\}$. By our observation, the unique pair of vertices mapping to $(i, i + j)$ is (v_{ti}, v_{ti+j}) . In particular, this is a pair of nonadjacent vertices (in $G_{k,qt+1}$) since $|tj| < qt + 1$. Thus the pre-image of any non-adjacent pair in $G_{k,q+1}$ is a unique pair of non-adjacent vertices in $G_{k,qt+1}$. Hence, φ is a homomorphism.

Next we show φ is uniquely extendible. Let v_{ti} be any arbitrary vertex in $V(G_{k,qt+1})$. Suppose to the contrary that φ is not uniquely extendible at v_{ti} . By Proposition 2 there is a vertex u in $\bigcap_{w \in N(v_{ti})} N(\varphi(w))$ different from i . Since $\varphi(w) \notin N(\varphi(w))$, it must be the case that $u \neq \varphi(w)$ for any $w \in N(v_{ti})$. Hence, $u = \varphi(v)$ for some $v \notin N(v_{ti})$, i.e. $v \in \{v_{ti \pm 1}, v_{ti \pm 2}, \dots, v_{ti \pm qt}\}$. Thus assume $u = i + jg$ for some $1 \leq j \leq qt$. (The case $-qt \leq j \leq -1$ is analogous.) We now establish a neighbour of v_{ti} is mapped to a nonneighbour of u , contrary to the definition of u . Observe, $tq + 1 \leq tq + j \leq 2tq$. Since $k \geq 3tq + 1$, v_{ti} is adjacent to $v_{ti+tq+j}$. Also, $\varphi(v_{ti+tq+j}) = i + q + jg \pmod{k}$. By assumption $u \in N(\varphi(v_{ti+tq+j}))$; thus, $u = i + jg$ is adjacent to $\varphi(v_{ti+tq+j}) = i + q + jg$ (in $G_{k,q+1}$), a contradiction. \square

By Theorem 4 above we obtain an infinite family of counterexamples to the question posed by Albertson and West.

Corollary 5. Given any positive integer ℓ , there exist positive integers k, q and k', q' with $\frac{k'}{q'} - \frac{k}{q} \geq \ell$ for which the following holds: for any positive integer d , there is a graph G with circular chromatic number k/q , and which has two circular cliques $G_{k,q}$ at distance d such that some (k', q') -precolouring of the two cliques is nonextendible.

The reader will undoubtedly notice that in the corollary above, $k' = k$. We now turn our attention to constructing (k, q) -colourable graphs that have nonextendible (k', q') -precolourings where $k' \neq k$. Continuing with the idea from the proof above, we study homomorphisms of the form $\varphi(v_i) = gi \pmod{k}$.

Definition 3. Let $\varphi : G_{k,q} \rightarrow G_{k',q'}$ be a homomorphism. Label $V(G_{k,q}) = \{v_0, v_1, \dots, v_{k-1}\}$. If $\varphi(v_i) = gi \pmod{k'}$ for all $v_i \in V(G_{k,q})$, then we say g generates φ .

Note that g acts on the vertices in $G_{k',q'}$. Hence, it makes sense to study mappings of this form for a fixed g and (k', q') , but differing input graphs, i.e. differing (k, q) .

Theorem 6. Suppose g generates $\varphi : G_{k,q} \rightarrow G_{k',q'}$ which is uniquely extendible. Then for any integer $t \geq 1$, g generates $\varphi' : G_{k+2tk',q+tk'} \rightarrow G_{k',q'}$ which is also uniquely extendible

Proof. Let $V(G_{k,q}) = \{v_0, v_1, \dots, v_{k-1}\}$, and $V(G_{k',q'}) = \{0, 1, \dots, k' - 1\}$ as above. Also, let $V(G_{k+2tk',q+tk'}) = \{u_0, u_1, \dots, u_{k+2tk'-1}\}$.

Let $v_i \in V(G_{k,q})$. Since φ is uniquely extendible we have

$$\left| \bigcap_{w \in N_{G_{k,q}}(v_i)} N_{G_{k',q'}}(\varphi(w)) \right| = \left| \bigcap_{j=i+q}^{i+k-q} N_{G_{k',q'}}(\varphi(v_j)) \right| = \left| \bigcap_{j=i+q}^{i+k-q} N_{G_{k',q'}}(g \cdot j) \right| = 1$$

where the product $g \cdot j$ is reduced modulo k' . Since φ is a homomorphism, $\varphi(v_i) = gi$ is the unique vertex in $\cap_{j=i+q}^{i+k-q} N_{G_{k',q'}}(g \cdot j)$, a fact we use below.

Now consider $u_i \in V(G_{k+2tk',q+tk'})$. Under φ'

$$\begin{aligned} \left| \bigcap_{w \in N_{G_{k+2tk',q+tk'}}(u_i)} N_{G_{k',q'}}(\varphi(w)) \right| &= \left| \bigcap_{j=i+q+tk'}^{i+k+2tk'-q-tk'} N_{G_{k',q'}}(\varphi(u_j)) \right| \\ &= \left| \bigcap_{j=i+q+tk'}^{i+k+2tk'-q-tk'} N_{G_{k',q'}}(g \cdot j) \right|. \end{aligned}$$

Again $g \cdot j$ is reduced modulo k' from which we obtain,

$$\left| \bigcap_{j=i+q+tk'}^{i+k+2tk'-q-tk'} N_{G_{k',q'}}(g \cdot j) \right| = \left| \bigcap_{j=i+q}^{i+k-q} N_{G_{k',q'}}(g \cdot j) \right| = 1.$$

By our remark above, $\varphi'(u_i) = \varphi(v_i) = gi$ is the unique vertex belonging to $\cap_{j=i+q}^{i+k-q} N_{G_{k',q'}}(g \cdot j)$. Hence φ' is a homomorphism and is uniquely extendible by Proposition 2. \square

Corollary 7. Given k', q' with $\frac{k'}{q'} \geq 2$, for any $\varepsilon > 0$ there exists k, q with $\varepsilon > \frac{k}{q} - 2 \geq 0$ and a uniquely extendible homomorphism $\varphi : G_{k,q} \rightarrow G_{k',q'}$.

Proof. Clearly, the identity map $G_{k',q'} \rightarrow G_{k',q'}$ is uniquely extendible and generated by 1. By Theorem 6 we may choose $k = k' + 2tk'$ and $q = q' + tk'$ for sufficiently large t . \square

Thus, Corollary 7 shows that for any $\frac{k'}{q'} \geq 2$, there are integers k and q such that:

- $3 > \frac{k}{q} \geq 2$;
- there is a (k', q') -precolouring of $P = G_{k,q}^1 \cup G_{k,q}^n$ in $G_{k,q} \bowtie P_n$; and,
- this precolouring is not extendible (for any n).

Hence, we obtain an infinite family of counterexamples to the Albertson and West conjecture where $k' \neq k$. However, in our examples $k' \leq k$. Given the positive results published for (ordinary) precolourings of (ordinary) cliques and the use of Kempe chains, it does seem reasonable to study the case when $k' > k$.

In the next section we present positive extension results; however, either the cliques or the colourings are classical cliques or colourings. We conclude this section with a small result highlighting the necessity of the hypothesis $k > 3(q - 1)$ (or equivalently $k - 2q + 2 > q - 1$) appearing in the forthcoming theorems.

Corollary 8. If $k - 2q + 2 \mid q - 1$, then there exists a uniquely extendible homomorphism $\varphi : G_{k,q} \rightarrow G_{k-2q+2,1}$.

Proof. Let $q - 1 = t(k - 2q + 2)$. The identity map $G_{k-2q+2,1} \rightarrow G_{k-2q+2,1}$ is uniquely extendible and generated by 1. By Theorem 6 there is a uniquely extendible homomorphism $\varphi : G_{(k-2q+2)+2t(k-2q+1),1+t(k-2q+2)} \rightarrow G_{k-2q+1,1}$ or $\varphi : G_{k,q} \rightarrow G_{k-2q+1,1}$. \square

3. Positive results

In [2] Albertson and Moore study the problem of extending a $(k + 1)$ -colouring of a k -colourable graph where the precoloured components are k -cliques. They also study the problem when the precoloured components are general subgraphs. In the latter case the penalty for having general subgraphs is a larger number of colours may be required for the extension. In this spirit we now turn attention to extending a (k', q') -colouring of a (k, q) -colourable graph where the precoloured components are circular cliques. As the results in the previous section indicate, this general situation is complex. Thus we restrict our attention to two special cases: either $q = 1$ as in Theorem 10 or $q' = 1$ as in Theorem 12. In an approach similar to Albertson and Moore, we reduce to the case where the precoloured components are essentially cliques. Thus the following theorem will be of particular use.

Theorem 9 (Albertson and Moore [2]). Let G be a graph with $\chi(G) = k$ and $P \subseteq V(G)$ such that the components of $G[P]$ are isomorphic to K_k . If $d(P) \geq 2k + 2\lfloor \frac{k}{2} \rfloor$, then any $(k + 1)$ -colouring of P extends to a $(k + 1)$ -colouring of G .

Our first result is on circular colourings of k -cliques. Notice for $\frac{k'}{q'} \geq k + 1$, the vertices $0, q', 2q', \dots, kq'$ induce a copy of K_{k+1} in $G_{k',q'}$. Our approach in the next theorem is to reduce an arbitrary (k', q') -precolouring to a $(k + 1)$ -colouring using multiples of q' . At this point we use Theorem 9 to extend.

$V(G_{k,q})$	I_0			I_1			I_2			I_3			I_4	
	0	1	2	3	4	5	6	7	8	9	10	11	12	13
$i = 1$	0	0	1	1	2	2	2	3	3	3	4	4	4	0
$i = 2$	0	0	1	1	2	2	5	3	3	3	4	4	4	0
$i = 3$	0	0	1	2	2	2	5	3	3	3	4	4	4	0
$i = 4$	1	1	1	2	2	2	5	5	5	3	4	4	4	0
$i = 5$	1	1	1	2	2	2	5	5	5	3	3	3	4	0
$i = 6$	1	1	1	2	2	2	5	5	5	3	3	3	4	4

Fig. 2. A 6-colouring of $G_{14,3} \bowtie P_6$ modified from $i = 1$ to $i = 6$ into a canonical colouring. Changes at each round are marked in bold. For $i = 2$ the interval $[l_2, r_2]$ is partitioned to avoid a multiple of $q = 3$. For $i = 3$, the vertices in I_1 are coloured 2. For $i = 4$, the intervals I_0, I_2 are made monochromatic. For $i = 5, i = 6$, the intervals I_3, I_4 are made respectively monochromatic.

Theorem 10. Let G be a k -colourable graph and $P \subseteq V(G)$ such that P induces k -cliques. If $\frac{k'}{q'} \geq k + 1$ and $d(P) \geq 4k + 2\lfloor \frac{k}{2} \rfloor$, then any (k', q') -colouring of P extends to a (k', q') -colouring of G .

Proof. Let φ be the (k', q') -colouring we wish to extend and let c be a k -colouring of G . Further suppose K is a clique from P . We shall extend φ to a ball of radius k around K so that all vertices on the boundary of the ball receive a colour that is a multiple of q' .

Given the symmetry of $G_{k',q'}$ we may assume for ease of demonstration one vertex receives colour from $\{0, 1, \dots, q' - 1\}$ and no vertex receives a colour from $\{k' - q', k' - q' + 1, \dots, k' - 1\}$ under φ . We label the vertices of K as v_0, v_1, \dots, v_{k-1} so that $\varphi(v_0) < \varphi(v_1) < \dots < \varphi(v_{k-1})$ and we designate the colour of v_i under c to be i , i.e. $c(v_i) = i$. Observe $c(v_i)q' \leq \varphi(v_i)$.

We now extend φ to a ball of radius k around K as follows. Let u be a vertex at distance i from K , $1 \leq i \leq k$. Let v be the vertex in K such that $c(u) = c(v)$. Assign

$$\varphi(u) = \begin{cases} jq' & \text{if } c(v) = j \text{ and } j < i \\ \varphi(v) & \text{otherwise.} \end{cases}$$

At this point φ colours the vertices at distance k from K with the colours $0, q', \dots, (k - 1)q'$.

To see this extension of φ is a homomorphism, consider a vertex u at distance i from K . Let w be a neighbour of u at distance at most i from K . There are vertices u', w' in K such that $c(u') = c(u)$ and $c(w') = c(w)$. First suppose $c(u) \geq i$. If $c(w) \geq i$ as well, we have $\varphi(w) = \varphi(w')$ and $\varphi(u)\varphi(w) = \varphi(u')\varphi(w') \in E(G_{k',q'})$. If $c(w) < i$, then

$$\varphi(w) = c(w)q' \leq (i - 1)q' \leq (c(u) - 1)q' \leq \varphi(u) - q'$$

and again $\varphi(u)\varphi(w)$ is an edge of $G_{k',q'}$. Next suppose $c(u) < i$ and thus $\varphi(u) = c(u)q'$. If $\varphi(w) = c(w)q'$, then $1 \leq |c(u) - c(w)| \leq k - 1$ implies $\varphi(u)\varphi(w)$ is an edge of $G_{k',q'}$. Finally if $c(w) \neq c(w)q'$, then $c(w) \geq i - 1$. Since $c(u) < i$ and $c(u) \neq c(w)$ we conclude

$$\varphi(u) = c(u)q' \leq (c(w) - 1)q' \leq \varphi(w) - q'.$$

Repeat this process for each clique in P obtaining a collection of precoloured components whose boundaries are coloured with $0, q', \dots, (k' - 1)q'$. The components are a distance of at least $2k + 2\lfloor \frac{k}{2} \rfloor$ apart. By Theorem 9 this colouring may be extended to the entire graph. \square

We now consider extending (classical) k' -colourings where the precoloured components are $G_{k,q}$. The general problem of extending colourings where the precoloured components are not cliques is considered in [2]. In our work the assumption that the precoloured components are circular cliques allows us to use fewer colours than is required in [2].

In the following we will consider the vertices of $G_{k,q}$ to be $\{0, 1, 2, \dots, k - 1\}$ and assume that all arithmetic is done modulo k . We think of the vertices $i, i + 1, i + 2, \dots, i + t$ as being *consecutive* and forming the interval $[i, i + t]$. Given $G_{k,q}$ with $k > 3(q - 1)$ we note that any independent set must be contained in an interval of q vertices (and thus of length $q - 1$). We also remind the reader that $c : G_{k,q} \rightarrow K_{\lceil k/q \rceil}$, where $c(i) = \lfloor i/q \rfloor$, is a homomorphism. The fibres of c are the intervals $[0, q - 1], [q, 2q - 1], [2q, 3q - 1], \dots, [(\lceil k/q \rceil - 2)q, (\lceil k/q \rceil - 2)q + q - 1], [(\lceil k/q \rceil - 1)q, k - 1]$. We label these intervals as $I_0, I_1, I_2, \dots, I_{\lceil k/q \rceil - 1}$. Each interval has q vertices with the possible exception of the last interval which has at most q vertices.

Any homomorphism $c' : G_{k,q} \rightarrow K_{k'}$ where the fibres of c' are the same as the fibres listed above for c is called a *canonical colouring* of $G_{k,q}$. Our strategy for extending a k' -colouring to all of G is to extend the precolouring of each copy of $G_{k,q}$ to a canonical colouring on the boundary vertices of a ball around the circular clique. Once this has been done we apply Albertson's method to extend the colouring to the remainder of the graph. The ideas in the proof are straightforward but rather technical. The reader may wish to consult Fig. 2 as a guide to the proof.

Lemma 11. Suppose $k, q \geq 2$ are integers such that $k > 3(q - 1)$. Further suppose $k' \geq \lceil \frac{k}{q} \rceil + 1$ and $n = \lceil \frac{k}{q} \rceil + 2$. Then any k' -colouring of $G_{k,q}^1$ in $G = G_{k,q} \bowtie P_n$ can be extended to all of G in such a way that $G_{k,q}^n$ receives a canonical colouring.

Proof. As in the previous section, we let the copies of $G_{k,q}$ in the product be denoted $G_{k,q}^i$ for $1 \leq i \leq n$. Also, $V(G_{k,q}^i) = \{(0, i), (1, i), \dots, (k-1, i)\}$. Let c_1 be the k' -colouring of $G_{k,q}^1$. We shall define for each $i = 2, 3, \dots, n$, $c_i : G_{k,q}^i \rightarrow K_{k'}$ so that c_{i-1} and c_i are compatible homomorphisms (in the sense of the collection defining a homomorphism $G_{k,q} \bowtie P_n \rightarrow K_{k'}$). We shall also denote the colours of $K_{k'}$ as $0, 1, 2, \dots, k' - 1$ where there is no risk of confusion based on the context.

Denote the colour of $(0, 1)$ under c_1 as 0 , i.e. let $c_1(0, 1) = 0$. All vertices in $G_{k,q}^1$ coloured 0 belong to an interval say $[(\ell_0, 1), (r_0, 1)]$ of length at most $q - 1$. Let $\ell_1 = r_0 + 1$ and denote by 1 the colour of $(\ell_1, 1)$, i.e. $c_1(\ell_1, 1) = 1$. Let r_1 be the last vertex in $[\ell_1, \ell_1 + q - 1]$ coloured 1 by c_1 . Continuing in this manner, we partition $G_{k,q}^1$ into intervals $[(\ell_0, 1), (r_0, 1)], [(\ell_1, 1), (r_1, 1)], [(\ell_2, 1), (r_2, 1)], \dots, [(\ell_m, 1), (r_m, 1)]$. This partition has the property that $c_1(\ell_i, 1) = i$ and all vertices of colour i lie in the interval $[(r_i - q + 1, 1), (r_i, 1)]$.

We now define c_2 with two goals in mind. First, the intervals identified above, $[\ell_j, r_j]$, will be monochromatic under c_2 . Second, we will ensure that one of these intervals does not contain a vertex of the form $(t \cdot q, 2)$.

If each interval $[(\ell_j, 1), (r_j, 1)]$ contains a vertex of the form $(t \cdot q, 1)$, then there are precisely $\lceil k/q \rceil$ such intervals. Since $k' \geq \lceil k/q \rceil + 1$, the colour $m + 1 \leq k'$ is not mapped to by c_1 . In this case, we find an interval $[(\ell_p, 1), (r_p, 1)]$ of length at least two, and partition the interval into two intervals $[(\ell_p, 1), (r'_p, 1)], [(r'_p + 1, 1), (r_p, 1)] = [\ell_{m+1}, r_{m+1}]$ where only one of the two intervals contains a vertex of the form $(i \cdot q, 1)$. Finally, relabel r'_p as r_p .

Define

$$c_2(i, 2) = j \quad \text{if } (i, 1) \in [(\ell_j, 1), (r_j, 1)], \quad 1 \leq j \leq m + 1.$$

On the other hand, if some interval $[(\ell_j, 1), (r_j, 1)]$ does not contain a vertex of the form $(t \cdot q, 1)$, then we define

$$c_2(i, 2) = j \quad \text{if } (i, 1) \in [(\ell_j, 1), (r_j, 1)], \quad 1 \leq j \leq m.$$

Hence at this point we have a colouring where all the vertices of colour j under c_2 are precisely the vertices in the interval $[(\ell_j, 2), (r_j, 2)]$. Also by construction there is an interval $[(\ell_s, 2), (r_s, 2)]$ not containing a vertex of the form $(t \cdot q, 2)$.

Let $t \cdot q$ be the (unique) multiple of q in $[(\ell_s - q + 1, 2), (\ell_s - 1, 2)]$.

Define

$$c_3(i, 3) = \begin{cases} s & \text{if } (i, 2) \in [(tq, 2), ((t+1)q - 1, 2)] \\ c_2(i, 2) & \text{otherwise.} \end{cases}$$

It is straightforward to verify c_3 is a proper colouring and compatible with c_2 . Recall $I_t = [t \cdot q, \min\{(t+1)q, k-1\}]$. In particular, a vertex belongs to interval I_t (in $G_{k,q}^3$) if and only if the vertex receives colour s under c_3 . In other words, we have stretched the interval of vertices receiving colour s to be precisely one of the canonical intervals.

For the moment assume both I_{t-1} and I_{t+1} have length $q - 1$. Let $c_3(tq - 1, 3) = j_1$ and $c_3((t+1)q, 3) = j_2$. Observe the vertices coloured j_1 by c_3 are all contained in the interval $[(t-1)q, 3), (tq - 1, 3)]$ and the vertices coloured j_2 are contained in the interval $[(t+1)q, 3), ((t+1)q - 1, 3)]$. From this observation we can define

$$c_4(i, 4) = \begin{cases} j_1 & \text{if } (i, 3) \in [((t-1)q, 3), (tq - 1, 3)] \\ j_2 & \text{if } (i, 3) \in [((t+1)q, 3), ((t+1)q - 1, 3)] \\ c_3(i, 3) & \text{otherwise.} \end{cases}$$

At this point canonical intervals I_{t-1}, I_t, I_{t+1} in $G_{k,q}^4$ are monochromatic.

The correctness of this extension depends on the fact that I_{t-1} has length $q - 1$ and thus all vertices of colour j_1 in $G_{k,q}^3$ belong to I_{t-1} . This argument may fail when working with $I_{\lceil k/q \rceil - 1}$. Hence we can continue extending outward from I_t either in two directions (as described above) or in one direction so that the final interval considered is $I_{\lceil k/q \rceil - 1}$. In the worst case ($I_t = I_0$ or $I_t = I_{\lceil k/q \rceil - 2}$), we require $n = \lceil k/q \rceil + 2$ steps to reach a canonical colouring. Should we reach a canonical colouring sooner, we simply fix that colouring for the remaining columns in the product. The result follows. \square

We finish the paper with an extension result for $(\lceil k/q \rceil + 1)$ -colourings of $G_{k,q}$ cliques in (k, q) -colourable graphs. A key observation is the following: In Section 2 the extension product is used to produce non-extendible examples. The following theorem demonstrates that knowing how to extend on the extension product allows one to extend colourings in general graphs. That is, under suitable conditions, the ability or non-ability to extend on the product $G_{k,q} \bowtie P_n$ tells us about the general case.

Theorem 12. Suppose $\chi_c(G) = \frac{k}{q}$ and $P \subseteq V(G)$ induces of a subgraph of G consisting of circular cliques $G_{k,q}$. Further suppose $k > 3(q-1)$ and $\gcd(k, q) = 1$. If the distance between any two cliques is at least $5\lceil k/q \rceil + 2$, then any $(\lceil k/q \rceil + 1)$ -colouring of P can be extended to a $(\lceil k/q \rceil + 1)$ -colouring of G .

Proof. Let K be a circular clique in P . Let $N_i(K)$ be the vertices at distance i from K . Let h be a (k, q) -colouring of G and let c be the $k' = (\lceil k/q \rceil + 1)$ -colouring of P .

Define $c_1 = c(K)$ and $h_1 = h(K)$. Since $\gcd(k, q) = 1$, h_1 must be an isomorphism, [5]; hence, $c' = c_1 \circ h_1^{-1}$ is a k' -colouring of $G_{k,q}$. By Lemma 11 there is an extension of c' to a canonical colouring of $G_{k,q}^n$ in the product $G_{k,q} \bowtie P_n$ where $n = \lceil k/q \rceil + 2$. Consider the subgraph $N_i(K)$. The (k, q) -colouring h restricted to $N_i(K)$ is a mapping to $G_{k,q}$. Composing this with c_{i+1}

gives a k' -colouring of $N_i(K)$. In particular we obtain an extension of c from K to $N_1(K) \cup N_2(K) \cup N_3(K) \cup \dots \cup N_{n-1}(K)$ where c_n is a canonical k' -colouring of $N_{n-1}(K)$. Because c_n is a canonical colouring, the quotient of the map, obtained by identifying vertices in the same fibre, is a copy of $K_{\lceil k/q \rceil}$. Moreover, c_n naturally defines a k' -colouring of this clique. Replace $K \cup N_1(K) \cup N_2(K) \cup N_3(K) \cup \dots \cup N_{n-1}(K)$ with a copy of $K_{\lceil k/q \rceil}$. Repeat this for each component of $G[P]$.

At this point we have a $\lceil k/q \rceil$ -colourable graph, a collection of $\lceil k/q \rceil$ -cliques at distance at least $3\lceil k/q \rceil$, and a k' -colouring of the cliques. We can now apply Albertson and Moore's result ([Theorem 9](#)) to extend the k' -colouring to the remainder of the graph. \square

To recap the above strategy, the mapping $h : G \rightarrow G_{k,q}$ when restricted to $K \cup N_1(K) \cup N_2(K) \cup N_3(K) \cup \dots \cup N_{n-1}(K)$ may be viewed as a homomorphism to $G_{k,q} \bowtie P_n$. The mapping c' is a k' -colouring of $G_{k,q}^1$ in this product. The extension of c' across this product tells us how to extend c' to the ball of radius $n - 1$ around K , so that the vertices on the boundary of the ball use only $\lceil k/q \rceil$ colours.

We remark the condition $\gcd(k, q) = 1$ can be relaxed. One only needs that the colouring c_1 factors through h to obtain c' for each copy of $K \in P$. The necessity of this factoring can be seen from the following example. Consider the graph G constructed from $K = G_{6,2}$ on vertices $\{0, 1, \dots, 5\}$ together with a single vertex a . Join a to $\{0, 1, 4, 5\}$. The resulting graph is $(6, 2)$ -colourable: map $\{0, 1\}$ to 0; $\{2, 3, a\}$ to 2; and $\{4, 5\}$ to 4. The precolouring $c_1(0) = 0$; $c_1(1) = c_1(2) = 1$; $c_1(3) = c_1(4) = 2$; $c_1(5) = 3$ is a 4-colouring of $G_{6,2}$ that does not extend the rest of the graph, i.e. to a .

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